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General solution of reflection equation for eight-vertex SOS model

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Abstract. By using the general solution of the reflection equation for the eight-vertex model, a general solution of the reflection equation for the eight-vertex SOS model is obtained. We also discuss the solution of the reflection equation for the case of the RSOS model.

1. Introduction

It is now generally realized that the integrability of two-dimensional lattice models is a consequence of the Yang–Baxter equation (YBE) [1, 2], which is usually written as

$$R_{12}(u_1-u_2)R_{13}(u_1-u_3)R_{23}(u_2-u_3)=R_{23}(u_2-u_3)R_{13}(u_1-u_3)R_{12}(u_1-u_2) \qquad (1)$$

where the R-matrix is the Boltzmann weight for the vertex models in two-dimensional statistical mechanics. As usual, $R_{12}(u)$, $R_{13}(u)$ and $R_{23}(u)$ act in $C^n\otimes C^n\otimes C^n$ with $R_{12}(u)=R(u)\otimes 1$, $R_{23}(u)=1\otimes R(u)$, etc.

Recently, the integrable systems with non-trivial boundary conditions have been attracting a great deal of interest, which was initiated by Cherednik [3] and Sklyanin [4]. They introduced a systematic approach for handling vertex models with non-trivial boundaries, which involves the so-called reflection equation (RE):

$$R_{12}(u_1-u_2)\overset{1}{K}(u_1)R_{21}(u_1+u_2)\overset{2}{K}(u_2)=\overset{2}{K}(u_2)R_{12}(u_1+u_2)\overset{1}{K}(u_1)R_{21}(u_1-u_2) \qquad (2)$$

where $\overset{1}{K}(u)$ and $\overset{2}{K}(u)$ are matrices respectively acting in $C^n\otimes 1$ and $1\otimes C^n$, which determine the non-trivial boundary terms in Hamiltonian. In the case of the eight-vertex model, $n=2$.

It is well known that vertex models are equivalent to solid-on-solid (SOS) models. Pearce *et al* [5] first obtain the RE for interaction-round-a-face (IRF) models and give a solution for RSOS model with one parameter. In this paper, we will give a solution of RE for SOS model with 3 parameters. The paper is organized as follows. We first review the results obtained by Baxter, Faddev [6] and Jimbo [7]. From the eight-vertex R-matrix and Yang–Baxter equation, the star–triangle relation is obtained. Next, we find the RE for IRF models corresponding to the RE for vertex models. Then, by using the general solution of the RE for the eight-vertex model, the general solution of the RE for the SOS model is obtained. Finally, we discuss the solution of the RE for the case of the restricted SOS (RSOS) model.

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2. Description of the model and vertex–IRF correspondence

It is known that many of the exactly solved two-dimensional models in statistical mechanics are equivalent to special cases of the eight-vertex model [8]; on the other hand, the eight-vertex model is a special case of the Z_n Belavin model. In this paper, we will start from the R-matrix of the eight-vertex model, which is written as

$$R(u) = \begin{pmatrix} a(u) & 0 & 0 & d(u) \\ 0 & b(u) & c(u) & 0 \\ 0 & c(u) & b(u) & 0 \\ d(u) & 0 & 0 & a(u) \end{pmatrix} \tag{3}$$

where

$$\begin{aligned} a(u) &= \rho\theta(2\eta)\theta(u)H(u+2\eta) \\ b(u) &= \rho\theta(2\eta)H(u)\theta(u+2\eta) \\ c(u) &= \rho H(2\eta)\theta(u)\theta(u+2\eta) \\ d(u) &= \rho H(2\eta)H(u)H(u+2\eta) \\ \rho &= \theta^{-1}(0). \end{aligned} \tag{4}$$

Here $H(u)$ and $\theta(u)$ are Jacobi theta functions. Their definition and properties can be found in [5–7].

Next, let us introduce the set of notation used in this paper. We define

$$\begin{aligned} \tilde{Z}^{l+1,l}(u) &\equiv (-x_l^2(u), x_l^1(u)) \\ \tilde{Z}^{l-1,l}(u) &\equiv (y_l^2(u), -y_l^1(u)) \\ Z^{l-1,l}(u) &\equiv \begin{pmatrix} x_l^1(u) \\ x_l^2(u) \end{pmatrix} \\ Z^{l+1,l}(u) &\equiv \begin{pmatrix} y_l^1(u) \\ y_l^2(u) \end{pmatrix} \end{aligned} \tag{5}$$

where

$$x_l^1(u) = H(s+2l\eta-u) \qquad y_l^1(u) = \frac{1}{h(w_l)}H(t+2l\eta+u)$$

$$x_l^2(u) = \theta(s+2l\eta-u) \qquad y_l^2(u) = \frac{1}{h(w_l)}\theta(t+2l\eta+u).$$

l is an integer, s and t are arbitrary complex parameters, $h(u) = H(u)\theta(u)$, and $w_l = \frac{s+t}{2} + 2l\eta - K$. If we let $w_0 = \frac{s+t}{2} - K$, we can see that $w_l = w_0 + 2l\eta$. Here K is a half-period of $\theta(u)$, with $K'/K = -\mathrm{i}\tau$. One can easily check that

$$Z^{l-1,l}(u)\tilde{Z}^{l-1,l}(u) + Z^{l+1,l}(u)\tilde{Z}^{l+1,l}(u) = F(u)\cdot \mathrm{id}. \tag{6}$$

where

$$F(u) = x_l^1(u)y_l^2(u) - x_l^2(u)y_l^1(u) = \frac{2h(u+\frac{t-s}{2})}{h(K)}. \tag{7}$$

Explicitly, the function $F(u)$ is independent of l.

Using the properties of Jacobi theta-functions, one can find [5, 6] that

$$R(u-v)Z^{l+1,l}(u)\otimes Z^{l,l-1}(v)=h(u-v+2\eta)Z^{l,l-1}(u)\otimes Z^{l+1,l}(v) \tag{8}$$

$$R(u-v)Z^{l-1,l}(u)\otimes Z^{l,l+1}(v)=h(u-v+2\eta)Z^{l,l+1}(u)\otimes Z^{l-1,l}(v) \tag{9}$$

$$\begin{aligned}R(u-v)Z^{l+1,l}(u)\otimes Z^{l,l+1}(v)&=\frac{h(2\eta)h(w_{l+1}+u-v)}{h(w_{l+1})}Z^{l,l+1}(u)\otimes Z^{l+1,l}(v)\\&\quad+h(u-v)Z^{l+2,l+1}(u)\otimes Z^{l+1,l+2}(v)\end{aligned} \tag{10}$$

$$\begin{aligned}R(u-v)Z^{l-1,l}(u)\otimes Z^{l,l-1}(v)&=\frac{h(2\eta)h(w_{l-1}-u+v)}{h(w_{l-1})}Z^{l,l-1}(u)\otimes Z^{l-1,l}(v)\\&\quad+\frac{h(u-v)h(w_l)h(w_{l-2})}{h^2(w_{l-1})}Z^{l-2,l-1}(u)\otimes Z^{l-1,l-2}(v).\end{aligned} \tag{11}$$

We call an ordered pair (a,b) admissible if $|a-b|=1$ with a,b taking integers. Let

$$W\left(\begin{array}{cc} a & b \\ d & c \end{array}\middle| u\right)$$

denote the Boltzmann weight of the IRF model corresponding to a configuration a,b,c,d round a face. We say that

$$W\left(\begin{array}{cc} a & b \\ d & c \end{array}\middle| u\right)$$

is admissible if (a,b), (b,c), (a,d) and (d,c) are all admissible, the non-admissible weights are set to 0. $\tilde{Z}^{ab}(u)$ and $Z^{ab}(u)$ are not defined if (a,b) is not admissible. One can see that equations (8)–(11) can be written in the simple form

$$R(u-v)Z^{ab}(u)\otimes Z^{bc}(v)=\sum_d W\left(\begin{array}{cc} a & b \\ d & c \end{array}\middle| u-v\right)Z^{dc}(u)\otimes Z^{ad}(v). \tag{12}$$

The non-zero weights take the form

$$\begin{aligned}
W\left(\begin{array}{cc} l\pm1 & l \\ l & l\mp1 \end{array}\middle| u\right)&=h(u+2\eta)\\
W\left(\begin{array}{cc} l & l\pm1 \\ l\pm1 & l \end{array}\middle| u\right)&=\frac{h(2\eta)h(w_l\mp u)}{h(w_l)}\\
W\left(\begin{array}{cc} l & l-1 \\ l+1 & l \end{array}\middle| u\right)&=h(u)\\
W\left(\begin{array}{cc} l & l+1 \\ l-1 & l \end{array}\middle| u\right)&=h(u)\frac{h(w_{l+1})h(w_{l-1})}{h^2(w_l)}.
\end{aligned} \tag{13}$$

One can find the star–triangle relation for IRF models corresponding to the YBE for vertex models,. Multiplying both sides of the Yang–Baxter equation (1) from the left by $Z^{af}(u_1)\otimes Z^{fe}(u_2)\otimes Z^{ed}(u_3)$, with the help of (12), we get

$$\begin{aligned}
&\sum_{b,c,g}W\left(\begin{array}{cc} b & g \\ c & d \end{array}\middle| u_1-u_2\right)W\left(\begin{array}{cc} a & f \\ b & g \end{array}\middle| u_1-u_3\right)W\left(\begin{array}{cc} f & e \\ g & d \end{array}\middle| u_2-u_3\right)\\
&\qquad\times Z^{cd}(u_1)\otimes Z^{bc}(u_2)\otimes Z^{ab}(u_3)\\
&\quad=\sum_{b,c,g}W\left(\begin{array}{cc} a & g \\ b & c \end{array}\middle| u_2-u_3\right)W\left(\begin{array}{cc} g & e \\ c & d \end{array}\middle| u_1-u_3\right)W\left(\begin{array}{cc} a & f \\ g & e \end{array}\middle| u_1-u_2\right)
\end{aligned}$$

$$\times Z^{cd}(u_1) \otimes Z^{bc}(u_2) \otimes Z^{ab}(u_3). \tag{14}$$

Thus we are led to the star–triangle relation. Here we have used the argument that $Z^{l-1,l}(u)$ and $Z^{l+1,l}(u)$ are linearly independent [7]. We write the star–triangle relation explicitly as

$$\sum_g W\left(\begin{array}{cc} b & g \\ c & d \end{array}\middle| u_1-u_2\right) W\left(\begin{array}{cc} a & f \\ b & g \end{array}\middle| u_1-u_3\right) W\left(\begin{array}{cc} f & e \\ g & d \end{array}\middle| u_2-u_3\right)$$
$$= \sum_g W\left(\begin{array}{cc} a & g \\ b & c \end{array}\middle| u_2-u_3\right) W\left(\begin{array}{cc} g & e \\ c & d \end{array}\middle| u_1-u_3\right) W\left(\begin{array}{cc} a & f \\ g & e \end{array}\middle| u_1-u_2\right). \tag{15}$$

For simplicity, we use the following symmetrizability so that the face weights enjoy the properties:

$$W\left(\begin{array}{cc} a & b \\ d & c \end{array}\middle| u\right) = W\left(\begin{array}{cc} c & b \\ d & a \end{array}\middle| u\right) = W\left(\begin{array}{cc} a & d \\ b & c \end{array}\middle| u\right). \tag{16}$$

The symmetrizability of the face weights takes the form

$$W\left(\begin{array}{cc} a & b \\ d & c \end{array}\middle| u\right) \to \frac{\mathrm{i}^b}{\mathrm{i}^d}\frac{s(a,b)s(b,c)}{s(a,d)s(d,c)} W\left(\begin{array}{cc} a & b \\ d & c \end{array}\middle| u\right) \tag{17}$$

and demands that

$$\frac{s(l,l+1)s(l+1,l)}{s(l,l-1)s(l-1,l)} = \frac{h(w_l)}{\sqrt{h(w_{l-1})h(w_{l+1})}} \tag{18}$$

where $\mathrm{i} = \sqrt{-1}$. One can choose

$$s(l,l+1) = \sqrt{h(w_l)}$$
$$s(l,l-1) = \frac{1}{\sqrt{h(w_l)}}. \tag{19}$$

Thus, equation (18) can be fulfilled. Usually, the so-called Boltzmann weights for the IRF model refer to the symmetric face weights defined by (17). In the rest of this paper, we will also denote the symmetric face weights as W. One can distinguish them from each other from the context. By definition (17), the non-zero face weights are

$$W\left(\begin{array}{cc} l\pm 1 & l \\ l & l\mp 1 \end{array}\middle| u\right) = h(u+2\eta)$$
$$W\left(\begin{array}{cc} l & l\pm 1 \\ l\pm 1 & l \end{array}\middle| u\right) = \frac{h(2\eta)h(w_l \mp u)}{h(w_l)} \tag{20}$$
$$W\left(\begin{array}{cc} l & l\mp 1 \\ l\pm 1 & l \end{array}\middle| u\right) = -h(u)\frac{\sqrt{h(w_{l-1})h(w_{l+1})}}{h(w_l)}.$$

It can easily be checked that the star–triangle relation is unaffected under the symmetrizability (17).

3. The face RE and its general solution for the eight-vertex SOS model

Now, let us examine the case of the reflection equation (2). Firstly, we define

$$K(g,a;f|u) \equiv \tilde{Z}^{gf}(u)K(u)Z^{af}(-u). \tag{21}$$

Multiplying both sides of the RE from the left by $Z^{bc}(-u_1) \otimes Z^{ab}(-u_2)$, we have

$$\begin{aligned}\text{RHS} &= \overset{2}{K}(u_2)R_{12}(u_1+u_2)\overset{1}{K}(u_1)\{P_{12}R_{12}(u_1-u_2)P_{12}\}\{P_{12}Z^{ab}(-u_2)\otimes Z^{bc}(-u_1)\} \\ &= \overset{2}{K}(u_2)R_{12}(u_1+u_2)\overset{1}{K}(u_1)\{P_{12}R_{12}(-u_2-(-u_1))Z^{ab}(-u_2)\otimes Z^{bc}(-u_1)\} \\ &= \sum_f \overset{2}{K}(u_2)R_{12}(u_1+u_2)W\left(\begin{array}{cc} a & b \\ f & c \end{array}\middle| u_1-u_2\right)\{(K(u_1)Z^{af}(-u_1))\otimes Z^{fc}(-u_2)\}\end{aligned} \tag{22}$$

where equation (12) has been used. Then applying equations (6) and (21),

$$\begin{aligned}\cdots &= \sum_f \overset{2}{K}(u_2)R_{12}(u_1+u_2)W\left(\begin{array}{cc} a & b \\ f & c \end{array}\middle| u_1-u_2\right) \\ &\qquad \times\{\sum_g \frac{1}{F(u_1)}(Z^{gf}(u_1)\tilde{Z}^{gf}(u_1)K(u_1)Z^{af}(-u_1))\otimes Z^{fc}(-u_2)\} \\ &= \frac{1}{F(u_1)}\sum_{gf}\overset{2}{K}(u_2)R_{12}(u_1+u_2)Z^{gf}(u_1)\otimes Z^{fc}(-u_2) \\ &\qquad \times K(g,a;f|u_1)W\left(\begin{array}{cc} a & b \\ f & c \end{array}\middle| u_1-u_2\right) \\ &= \frac{1}{F(u_1)F(u_2)}\sum_{ed}\left\{\sum_{gf}K(e,g;d|u_2)W\left(\begin{array}{cc} g & f \\ d & c \end{array}\middle| u_1+u_2\right)K(g,a;f|u_1)\right. \\ &\qquad \left.\times W\left(\begin{array}{cc} a & b \\ f & c \end{array}\middle| u_1-u_2\right)\right\}Z^{dc}(u_1)\otimes Z^{ed}(u_2).\end{aligned} \tag{23}$$

Similarly

$$\begin{aligned}\text{LHS} &= \frac{1}{F(u_1)F(u_2)}\sum_{ed}\left\{\sum_{gf}W\left(\begin{array}{cc} e & f \\ d & c \end{array}\middle| u_1-u_2\right)K(e,g;f|u_1)W\left(\begin{array}{cc} g & b \\ f & c \end{array}\middle| u_1+u_2\right)\right. \\ &\qquad \left.\times K(g,a;b|u_2)\right\}Z^{dc}(u_1)\otimes Z^{ed}(u_2).\end{aligned} \tag{24}$$

By using the same argument, when we derive the star–triangle relation, we obtain the reflection equation for the IRF models:

$$\begin{aligned}&\sum_{gf}W\left(\begin{array}{cc} e & f \\ d & c \end{array}\middle| u_1-u_2\right)K(e,g;f|u_1)W\left(\begin{array}{cc} g & b \\ f & c \end{array}\middle| u_1+u_2\right)K(g,a;b|u_2) \\ &\qquad = \sum_{gf}K(e,g;d|u_2)W\left(\begin{array}{cc} g & f \\ d & c \end{array}\middle| u_1+u_2\right)K(g,a;f|u_1) \\ &\qquad \times W\left(\begin{array}{cc} a & b \\ f & c \end{array}\middle| u_1-u_2\right).\end{aligned} \tag{25}$$

Note that the face weights used in (25) do not have the symmetries (16), so we also need a symmetrizability. Let

$$K(a,b;c|u) \to \frac{s(b,c)}{s(a,c)}K(a,b;c|u) \tag{26}$$

in the same time, take the symmetrizability for face weights (17), we find that the face reflection equation (25) is unaffected. So, one can also use symmetric Boltzmann weights in (25).

In what follows, we will find the general solution of the RE for the eight-vertex SOS model, which determines the boundary conditions. It is known that several groups have already obtained general and less general solutions of the RE for eight-vertex model [9–12], so it is not necessary to calculate the solution of the face RE directly from (25). One can find it through the known results. We choose the general solution of the RE for the eight-vertex model [11] as

$$K(u) = 1 + C_1\frac{\tilde{H}(u)}{\tilde{\theta}(u)}\sigma_1 + C_2\frac{\tilde{H}(u)}{\tilde{\theta}(u+K)}\sigma_2 + C_3\frac{\tilde{H}(u)}{\tilde{H}(u+K)}\sigma_3 \tag{27}$$

with

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{28}$$

where C_i are arbitrary parameters with $i = 1, 2, 3$, here $K'/K = \frac{1}{2}i\tau$.

Now, we are led to the general solution of (25), which is written as

$$K(a, b; c|u) = \frac{s(b, c)}{s(a, c)}\tilde{Z}^{ac}(u)K(u)Z^{bc}(-u). \tag{29}$$

For the case of the RSOS model, the following relation should be satisfied:

$$\eta = \frac{K}{r} \qquad w_0 = 0 \tag{30}$$

and l is restricted to the integer values

$$l = 1, 2, \ldots, r-1. \tag{31}$$

Here r is an arbitrary integer not less than 3. Just like the star–triangle relation, equation (15) is satisfied by the RSOS model because that the terms corresponding to $g = 0, r$ are deleted, we have found that the face reflection equation is also satisfied by the RSOS model because that the terms corresponding to $g, f = 0, r$ are deleted, which is caused by the relation

$$h(w_0) = h(w_r) = 0. \tag{32}$$

Thus, in the case of $b \neq r, a \neq 0$, the solution (29) can also be applied to the RSOS model with (30), (31) being satisfied. What needs to be noted is that relation (29) is not a complete solution to the RE (25).

4. Discussions and summary

Sklyanin has shown that the 'double-row' transfer matrix with open boundary conditions can be written as

$$t(u) = tr K_+(u)T(u)K(u)T^{-1}(-u) \tag{33}$$

where $K(u)$ are solutions of the RE (2) and $K_+(u)$ are solutions of the dual RE. There is an isomorphism between $K_+(u)$ and $K(u)$ [4]. By using the unitarity and cross unitarity properties of the R-matrix which has the P- and T-symmetry satisfied by the six-vertex and eight-vertex models, with the help of the RE and dual RE one can prove that the 'double-row' transfer matrix constitutes a one-parameter commutative family

$$[t(u), t(v)] = 0 \tag{34}$$

which ensures the integrability of the system. Using the vertex–IRF correspondence, the elements of the 'double-row' transfer matrix for SOS model are defined as

$$D(u)=\sum_{c_0\cdots c_n} W\begin{pmatrix} c_0 & c_1 \\ a_0 & a_1 \end{pmatrix}\bigg|u\bigg)\cdots W\begin{pmatrix} c_{n-1} & c_n \\ a_{n-1} & a_n \end{pmatrix}\bigg|u\bigg) K(a_n,b_n;c_n|u)$$

$$\times K_+(a_0,b_0;c_0|u) W\begin{pmatrix} b_0 & b_1 \\ c_0 & c_1 \end{pmatrix}\bigg|u\bigg)\cdots W\begin{pmatrix} b_{n-1} & b_n \\ c_{n-1} & c_n \end{pmatrix}\bigg|u\bigg). \tag{35}$$

It is known that the face weights satisfy the inversion relations [7] and the star–triangle relations. Using the face RE (25) and dual face RE, which can be obtained in a manner similar to that for the dual RE for vertex models, one should prove that

$$[D(u), D(v)] = 0 \tag{36}$$

which ensures the integrability of the IRF models with open boundary conditions. Pearce *et al* have proved diagrammatically that $D(u)$ form a commuting family [5].

In conclusion, we have found the face RE for the IRF model, and obtained the general solution of the RE for the SOS model. We have seen that the face RE can also be applied to the IRF models corresponding to the Z_n Belavin model; the definition of admissible pair (a, b) and the face Boltzmann weights should be changed correspondingly. Though Cherednik and we have both obtained the solution of the RE for the Z_n Belavin model [3, 11, 13], the solution of the face RE cannot be obtained through those results, because until now we have not had the explicit $\tilde{Z}^{ab}(u)$ form. Since Sklyanin found the spectrum of the Hamiltonian with non-trivial boundary terms for the six-vertex model by the algebraic Bethe ansatz, an open challenge is that of how to solve this problem for the eight-vertex model. We know that the star–triangle relation provides the commutation relation for the eight-vertex model when we find the spectrum of the Hamiltonian with periodic boundary conditions; from this point of view, the face RE can also provide the commutation relation when we diagonalize the Hamiltonian with indepedent boundary conditions.

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